characteristic turbulence scale l. If the scale $l > (C_DA)^{1/2}$, then the energy builds up in a few scale lengths and then decays rather rapidly. If, however, $l < (C_DA)^{1/2}$, then the energy increase is very slight, and the decay is rather slow. For $l = 0.1(C_DA)^{1/2}$, the energy is practically constant. The asymptotic decay for $\xi \gg 1$ is, in every case, proportional to ξ^{-2}.

These results are not inconsistent with the currently accepted ideas about turbulent energy decay. The persistence of the large eddies is rather clearly brought out; whereas, at the same time, the small eddies show a tendency to hold on to their energy, as predicted by the Kolmogoroff hypothesis. The implications of the results for the calculation of the radar cross section of turbulent plasmas is the following: if we accept the hypothesis that

$$\langle n_e^2(\kappa_i, \xi) \rangle \propto \langle N_e(\xi)\rangle^2 E(\kappa_i, \xi)$$

where n_e are the fluctuations in electron density, $\langle N_e \rangle$ is the average electron density, and $E(\kappa_i, \xi)$ is the turbulent energy of eddies in the wave number interval $\Delta\kappa(\kappa_i)$, then it would appear that the Kolmogoroff assumption $E(\kappa_i, \xi) \approx$ const is quite acceptable for wavelengths that are rather small as compared with the characteristic dimensions of the body. For wavelengths which are longer than $(C_DA)^{1/2}$, it is probably necessary to take into account the fact that the turbulent energy is not constant.

References

[1] Menkes, J., "Scattering of radar waves by an underdense turbulent plasma," AIAA J. **2**, 1154–1156 (1964).

[2] Prandtl, L., "Über ein neues Formelsystem für die ausgebildete Turbulenz," *Gesammelte Abhandlungen* (Julius Springer-Verlag, Heidelberg, 1961), Vol. II, pp. 874–887.

[3] Bateman, H., "The decay of a simple eddy," National Advisory Committee for Aeronautics, Rept. 144 (1922).

[4] Townsend, A. A., *The Structure of Turbulent Shear Flow* (Cambridge University Press, Cambridge, England, 1956).

Combined Forced and Free Convection Channel Flows in Magnetohydrodynamics

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Nomenclature

B = induced magnetic field
B_0 = applied magnetic field
g = gravitational acceleration
i = $(-1)^{1/2}$
L = half-width of channel
p = pressure
T, T^* = temperature
T_{w0} = wall temperature at $z = 0$
v = velocity
α = thermal diffusivity
β = volumetric expansion coefficient
θ^* = temperature difference, $T^* - T_{w0}$
μ = magnetic permeability
ν = kinematic viscosity
ρ = density
σ = electrical conductivity

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Introduction

CONSIDERABLE work has been done recently on problems dealing with heat transfer in electrically conducting fluids in the presence of electromagnetic fields. An extensive review for problems of this type is given in Ref. 1. This note deals with the combined forced and free convection of an electrically conducting fluid flowing inside an open-ended vertical channel formed by two identical parallel plates in the presence of a uniform transverse magnetic field. Solutions are obtained for the case when the surface temperatures of the plates vary linearly along the vertical distance. Such configuration in the magnetic field-free case was discussed by Ostrach.[2,3]

When the vertical temperature gradient is taken to be negative (which is the case of heating from below), there appears a critical Rayleigh number at which the solution of the velocity becomes infinite. This was attributed[3] to the appearance of instability as it would occur for a horizontal layer heated from below. The influence of the magnetic field in the latter case has been studied extensively.[4,5] The conclusion is that the magnetic field has a stabilizing effect. In the present configuration of vertical parallel plates, it is found that the magnetic field has the same effect.

Basic Equations and Their Solutions

For the fully developed laminar flow in a uniform transverse magnetic field, the velocity and induced magnetic field have only a vertical component, and all of the physical quantities except temperature and pressure are independent of the vertical coordinate z. Furthermore, the temperature of the fluid can be expressed as

$$T = T^*(y) + Nz \tag{1}$$

where N is the vertical temperature gradient, and y is the horizontal coordinate normal to the plates.

Under the conditions stated, the continuity equation is identically satisfied, and the momentum equations in the y and z component are

$$(\partial p/\partial y) + (1/\mu)B(dB/dy) = 0 \tag{2}$$

$$\nu(d^2v/dy^2) + (1/\rho\mu)B_0(dB/dy) + g\beta(\theta^* + Nz) - (1/\rho)(\partial p/\partial z) = 0 \tag{3}$$

and the energy and magnetic induction equations reduce to

$$Nv = \alpha(d^2\theta^*/dy^2) \tag{4}$$

$$(d^2B/dy^2) + \sigma\mu B_0(dv/dy) = 0 \tag{5}$$

In the energy equation we have neglected the viscous and Joulean dissipations.

Integrating Eq. (2) with respect to y, it gives

$$p = (B^2/2\mu) + f(z) \tag{6}$$

where $f(z)$ is an arbitrary function of z. When we substitute p into Eq. (3) and rearrange terms, it becomes

$$\nu(d^2v/dy^2) + (1/\rho\mu)B_0(dB/dy) + g\beta\theta^* = (1/\rho)(df/dz) - g\beta Nz \tag{7}$$

Since the right-hand side of Eq. (7) is a function of z only, whereas the left-hand side is a function only of y, both must be equal to a constant C_1. Thus, we write Eq. (7) as

$$\nu(d^2v/dy^2) + (1/\rho\mu)B_0(dB/dy) + g\beta\theta^* = C_1 \tag{8}$$

The constant C_1 must be related to the physics of the problem. It could be determined from either the end conditions of pressure, to which the channel is subjected, or the end conditions by the mass flow in the channel.

We introduce dimensionless variables by the substitutions $\eta = (y/L)$, $u = (Lv/\alpha)$, $t = (\theta^*/NL)$, $b = (B/B_0)$, M = Hartmann number = $B_0L(\sigma/\rho\nu)^{1/2}$, Ra = Rayleigh number =

$(g\beta NL^4/\nu\alpha)$, and $P_m = \alpha\sigma\mu$, for which Eqs. (8), (4), and (5) become, respectively,

$$(M^2/P_m)(db/d\eta) + (d^2u/d\eta^2) + Rat = C_2 \quad (9)$$

$$u = (d^2t/d\eta^2) \quad (10)$$

$$(1/P_m)(d^2b/d\eta^2) + (du/d\eta) = 0 \quad (11)$$

with $C_2 = (C_1L^3/\alpha\nu)$. The boundary conditions of Eqs. (9–11) are such that $u = b = t = 0$ at $\eta = \pm 1$.

The system of Eqs. (9–11) can be written as a fourth-order differential equation of u by eliminating t and b, which is

$$(d^4u/d\eta^4) - M^2(d^2u/d\eta^2) + Rau = 0 \quad (12)$$

From this equation the solution of u satisfying the boundary conditions is easily determined. The corresponding solutions of t and b are obtained from Eqs. (10) and (11) after u is found. These solutions are

$$u = C_3[(\cosh k_1\eta/\cosh k_1) - (\cosh k_2\eta/\cosh k_2)] \quad (13)$$

$$t = C_3\{(1/k_1^2)[(\cos k_1\eta/\cosh k_1) - 1] - (1/k_2^2)[(\cosh k_2\eta/\cosh k_2) - 1]\} \quad (14)$$

$$b = P_mC_3[(1/k_1 \cosh k_1)(\eta \sinh k_1 - \sinh k_1\eta) - (1/k_2 \cosh k_2)(\eta \sinh k_2 - \sinh k_2\eta)] \quad (15)$$

where

$$k_1 = \{(M^2/2) + [(M^4/4) - Ra]^{1/2}\}^{1/2}$$

$$k_2 = \{(M^2/2) - [(M^4/4) - Ra]^{1/2}\}^{1/2}$$

and

$$C_3 = \{C_2/\mathrm{Ra}[(1/k_1^2) - (1/k_2^2)] + M^2[(\tanh k_1/k_1 - \tanh k_2/k_2)]\}$$

The nondimensional flow rate w and heat transfer h are then

$$w = \int_{-1}^{1} u d\eta = 2C_3 [(\tanh k_1/k_1) - (\tanh k_2/k_2)] \quad (16)$$

$$h = (dt/d\eta)_{\eta=1} = C_3[(\tanh k_1/k_1) - (\tanh k_2/k_2)] \quad (17)$$

Eqs. (13–15) show that the relative influence of the magnetic field on the velocity, temperature, and induced magnetic field distributions depends upon the ratio of Hartmann number and the fourth root of Rayleigh number. In Fig. 1 we present the velocity and temperature profiles for $Ra = 1$, $C_2 = 1$, and different values of M^2. It is seen that, when M^2 increases, the velocity decreases rapidly and, at the same time, produces a higher temperature in fluid, thus decreasing the heat transfer. Figure 2 shows the induced magnetic field (b/P_m) for same values of Ra, C_2, and M^2. It should be noted that b is the ratio of B to the external field B_0. As M^2 approaches zero, (b/P_m) approaches a limiting value; for large M^2, (b/P_m) decreases. Thus, a maximum value of B is attained for some value of M^2. It should also be noted that P_m is proportional to the magnetic Reynolds number, large values of P_m corresponding to large induced magnetic fields B.

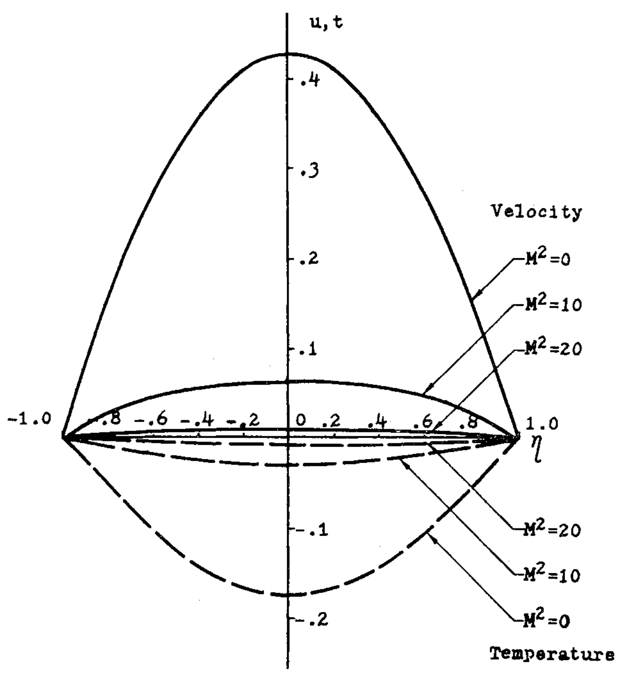

Fig. 1 Velocity and temperature distributions; *Ra* = 1, C_2 = 1, and *M*² = 0, 10, and 20.

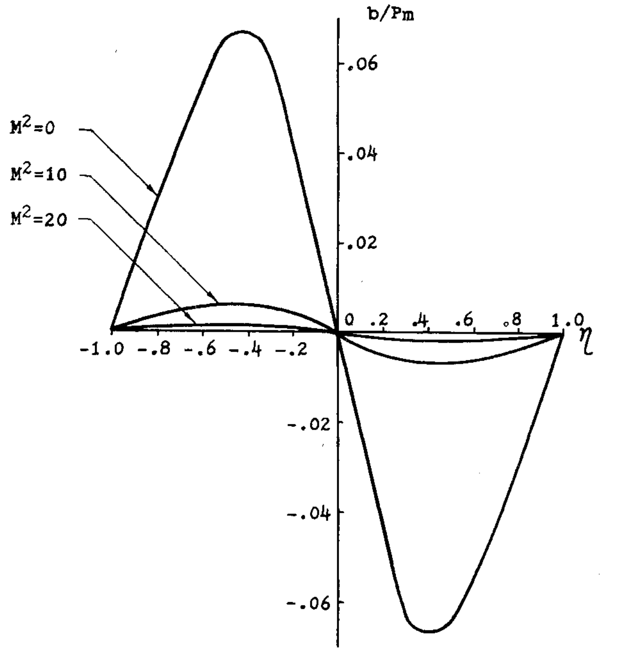

Fig. 2 Induced magnetic-field distributions; *Ra* = 1, C_2 = 1, and *M*² = 0, 10, and 20.

Unstable Conditions

The solutions of velocity, temperature, and induced magnetic field given by Eqs. (13–15) are bounded for all of the values of Ra and M^2 provided that $Ra > 0$. In the case of heating from below, since $N < 0$, we have $Ra < 0$. In this case k_2 becomes a pure imaginary number, and subsequently $\cosh k_2$ becomes equal to $\cos k_2'$ where $k_2' = ik_2$. Examination of the solutions of u, t, and b shows that there are critical values of k_2' (namely, $k_2' = n\pi/2$ where n is an integer), for which they become infinite. This is a result of the appearance of instability, as discussed in the magnetic field-free case. The critical Rayleigh number for the lowest eigenvalue $n = 1$ is

$$Ra = (\pi^2/4)[(\pi^2/4) + M^2] \quad (18)$$

At $M^2 = 0$, this value agrees with Ostrach's result; it increases linearly with the increase of M^2; i.e., the flow is stabilized by the magnetic field. This stabilizing effect is similar to those found in the case of horizontal layers heated from below.

References

[1] Romig, M. F., "The influence of electric and magnetic fields on heat transfer to electrically conducting fluids," Rand Corp. Research Memo. RM-3336-PR (1962).

[2] Ostrach, S., "Combined natural- and forced-convection laminar flow and heat transfer of fluids with and without heat sources in channels with linearly varying wall temperatures," NACA TN3141 (1954).

[3] Ostrach, S., "On the Flow, Heat Transfer, and Stability of Viscous Fluids Subjected to Body Forces and Heated from Below in Vertical Channels," *50 Jahre Grenzschichtforschung* (Friedrich Vieweg and Sohn, Braunschweig, Germany, 1955).

[4] Thompson, W. B., "Thermal convection in a magnetic field," Phil. Mag. **42**, 1417–1432 (1951).

[5] Chandrasckhar, S., *Hydrodynamic and Hydromagnetic Stability* (Clarendon Press, Oxford, England, 1961), Chap. IV.

Approximate Solutions of the Membrane Flutter Problem

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The linearised dynamic equation associated with two-dimensional membrane flutter in supersonic flow may be written (using the "static" aerodynamic approximation) as

$$\frac{\partial^2 Y}{\partial x^2} - \alpha \frac{\partial Y}{\partial x} - \gamma \frac{\partial^2 Y}{\partial t^2} = 0 \qquad (1)$$

where α and γ are constants.

Separation of variables by writing $Y = y(x)e^{i\omega t}$ gives

$$\frac{d^2y}{dx^2} - \alpha \frac{dy}{dx} + \lambda y = 0 \qquad (2)$$

It is now necessary to find eigenvalues $\lambda = \gamma\omega^2$ for a nontrivial solution to Eq. (2) subject to the boundary conditions $y(0) = y(1) = 0$. Exact solution leads to the result

$$\lambda = n^2\pi^2 + (\alpha^2/4) \qquad (3)$$

The question of approximate solution of this problem using the Galerkin method has raised doubts about the validity of this method with a non-self-adjoint second-order equation. It has been shown that the second and third terms in an approximating series lead to critical values of α at which two eigenvalues coalesce to form complex conjugates. The exact solution shows this flutter condition is spurious. However, the work of Keldysh[1] and Petrov[2] indicates that the exact eigenvalues of Eq. (2) can be obtained as the limit of a sequence of approximate eigenvalues (from the Galerkin method) formed by increasing the number of terms in the approximating series. It might be argued then that an examination of the effect of increasing the number of terms in the series on the critical value of α may show the spurious nature of the flutter boundary.

A Galerkin solution using the approximating series

$$y = \sum_{n=1}^{\infty} A_n \sin n\pi x \qquad (4)$$

in Eq. (2) leads to a matrix problem in which it is necessary to find the eigenvalues λ of the matrix $\{a_{mn}\}$ where

$$a_{nn} = n^2\pi^2$$

$$a_{mn} = \frac{2mn}{(m^2 - n^2)}\alpha[1 - (-)^{m+n}] \qquad m \neq n \qquad (5)$$

Since this matrix is not symmetric, it is quite possible that for some values of α the eigenvalues have nonzero imaginary parts. The eigenvalues of increasing order matrices have been evaluated and the critical value of α determined by the criterion that at flutter two of the eigenvalues will coalesce to form a complex conjugate pair.

Table 1 gives the number of modes and the critical value of α for each. If a greater number of terms were taken, α_{crit} would form a more obviously diverging sequence. An important feature of the solutions is that of instability occurring when the two highest eigenvalues coalesce. However, since only the lower eigenvalues have been determined with any accuracy, it is obvious at any stage that more terms must be taken to check the instability; as Bisplinghoff[3] suggests, an infinite number of terms are in fact needed. This leads to the conclusion that, although a finite α_{crit} has been obtained, it has no meaning because of the manner in which the instability occurs.

It is well known that Galerkin's method applied to a non-self-adjoint differential equation need have no relationship to the Ritz method applied to a variational problem. However, in the case of a second-order differential equation, it is always possible to introduce a multiplying factor that will convert the equation to the self-adjoint form. For Eq. (2), it is $e^{-\alpha x}$, giving the self-adjoint equation,

$$d/dx[e^{-\alpha x}(dy/dx)] + \lambda e^{-\alpha x}y = 0 \qquad (6)$$

(a particular case of the homogeneous Sturm-Liouville differential equation).

All second-order self-adjoint differential equations may be regarded as Euler equations arising from variational problems with homogeneous quadratic integrands. In this case, Eq. (6) is the Euler equation for the minimization of the integral

$$\int_0^1 e^{-\alpha x}\left\{\left(\frac{dy}{dx}\right)^2 - \lambda y^2\right\} dx \qquad (7)$$

In general, Galerkin's method applied to a self-adjoint equation leads to an eigenvalue problem involving symmetric matrices. When this technique is applied to Eq. (6) with the approximating series used previously, it leads to an eigenvalue problem of the form

$$|B - \lambda C| = 0 \qquad (8)$$

where B and C are symmetric matrices whose elements are given as

$$b_{mn} = \frac{[1 - (-)^{m+n}e^{-\alpha}][(m^2 + n^2)\pi^2 + \alpha^2]}{[(m-n)^2\pi^2 + \alpha^2][(m+n)^2\pi^2 + \alpha^2]}$$

$$c_{mn} = \frac{[1 - (-)^{m+n}e^{-\alpha}]}{[(m-n)^2\pi^2 + \alpha^2][(m+n)^2\pi^2 + \alpha^2]}$$

If the matrix C is positive definite, the eigenvalues of Eq. (8) are real, and Galerkin's method would lead directly to the correct result. Even a two-term solution gives

$$2\lambda = 5\pi^2 + \alpha^2 \pm \frac{3\pi^2(1 - e^{-\alpha})(\pi^2 + \alpha^2)(9\pi^2 + \alpha^2)}{\{(1 - e^{-\alpha})^2(\pi^2 + \alpha^2)^2(9\pi^2 + \alpha^2)^2 - \alpha^4(1 + e^{-\alpha})^2(4\pi^2 + \alpha^2)(16\pi^2 + \alpha^2)\}^{1/2}}$$

The denominator of the last term on the right-hand side is always real, hence a two-term Galerkin analysis of the self-adjoint equation indicates no instability.

Two conclusions may be drawn from this study: 1) the Galerkin method is applicable to the case of the supersonic membrane flutter problem and yields results in agreement with the exact solution when these results are interpreted correctly; any approximate solution should involve a convergence examination; and 2) if the dynamic equation is made self-adjoint, a two-term solution gives the correct answer without it being necessary to investigate the problem more deeply.

Finally it should be noted that it is rarely necessary to solve (exactly or approximately) any boundary-value prob-

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